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SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

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1. A DIFFERENTIAL equation of the first order is usually written in the form

$$\varphi(x, y, p) = 0, \quad \ldots \ldots \ldots \ldots \quad (1)$$

where $p = \dfrac{dy}{dx}$. Regarding x and y as the rectangular coordinates of a moving point, $p = \tan\varphi$, if φ denote the inclination of the point's motion to the axis of x. The relation (1) must therefore be regarded not as the equation of a curve, or relation between the coordinates of a point, but rather as a relation between the position of a moving point and the direction of its motion. Thus the point (x, y) may have any assumed position as (a, b), and the corresponding value of p is any one of the roots of

$$\varphi(a, b, p) = 0. \quad \ldots \ldots \ldots \ldots \quad (2)$$

Starting from the assumed position (a, b) the point may move in the direction assigned by one of the roots of (2); and, as the values of x and y vary, the value of p will in general vary; the point describing a curve. Different assumed initial positions will determine other curves, and we may say that a point satisfies the differential equation (1), provided it is moving in any one of the system of curves thus generated.

2. The general equation of this system of curves is the Complete Primitive of (1). This equation will contain but one arbitrary constant; since the condition to pass through a given point will determine the curve to be one of a limited number of curves of the system, this number being indicated by the degree of equation (1) with respect to p. The equation of the complete primitive will therefore be of the form,

$$f(x, y, c) = 0; \quad \ldots \ldots \ldots \ldots \quad (3)$$

and as the number of curves of the system (3) passing through a given point is indicated by the degree of the equation with respect to c, it is evident that p and c will occur in equations (1) and (3) respectively in the same degree.

3. Let us suppose that the system of curves represented by the complete primitive admits of an envelop: then a point moving in the envelop will always have the same direction as if it were moving in that one of the system of which it constitutes the point of contact with the envelop. Hence a point so moving will satisfy the differential equation; in other words the equation of the envelop or any branch of it will form a solution of eq. (1). Such a solution contains no arbitrary constant, and is called a Singular Solution.

4. If the differential equation is of the second degree with respect to p it may be written in the form

$$Ap^2 + Bp + C = 0, \quad \dots\dots\dots\dots \quad (4)$$

A, B and C denoting functions of x and y. In this case there will generally exist a certain region of the plane for which p is impossible, and a certain region for which p has two real and different values. While c passes through a complete cycle of values, the curve (3) sweeps twice over each point in the latter region, and the boundary between the two regions, where the values of p are equal, is the envelop of the system, that is, the curve whose equation is the singular solution. To find this solution we have then only to form the equation

$$B^2 - 4AC = 0, \quad \dots\dots\dots\dots\dots \quad (5)$$

the condition for equal roots in (4). Thus the differential equation being

$$xp^2 - py + a = 0$$

the singular solution is $\qquad y^2 = 4ax.$

5. It is to be observed however that we may in this manner obtain an equation which does not satisfy the given differential equation. The above reasoning, in fact, while it shows that the locus of (5) includes the envelop if there be one, it does not show that it can include no other branches. In the first place even when the branch in question is the boundary between the two regions mentioned in Art. 4, it may be the locus of a cusp in the complete primitive, and a point moving in such a locus will not generally satisfy the differential equation. In the second place the branch in question may not even form a portion of the boundary; for it may be the locus of the point of contact of two curves belonging to the system (3). As an example of the first case let the given equation be

$$ap^2 - py + x = 0.$$

Eq. (5) gives $\qquad y^2 - 4ax = 0,$

but it will be found on trial that this is not a solution of the given equation. The expression $y^2 - 4ax$ being negative within the parabola $y^2 = 4ax$ and positive outside of it, we conclude that it forms the boundary, and must therefore be the locus of a cusp of the complete primitive. As an example of the second case, given the equation

$$y^2p^2 + y^2 - a^2 = 0.$$

Eq. (5) gives
$$y^2(y^2 - a^2) = 0,$$
which includes the loci $y = 0$ and $y = \pm a$. The latter satisfy the given equation, but $y = 0$ does not. The complete primitive in this case is

$$y^2 + (x - c)^2 = a^2,$$

representing (since c is arbitrary) a circle whose radius is a, and whose centre moves on the axis of x; $y = \pm a$ constitute the envelop, and $y = 0$ is the locus of the point of contact of two circles of the system

6. The boundary line will necessarily be a part of the locus of eq'n (5), but it may happen that it is also one of the particular integrals included in the complete primitive. The following example from Boole's Differential Equations illustrates this, though the author employs a different criterion for the singular solutions. Given

$$p^2 - pxy + y^2 \log y = 0.$$

Eq. (5) gives
$$y^2(x^2 - 4 \log y) = 0;$$
the roots $y = 0$ and $x^2 = 4 \log y$ or $y = e^{\frac{1}{4}x^2}$ are both solutions of the given equation, and p is only possible in the region included between the curve $y = e^{\frac{1}{4}x^2}$ and the axis of x. The complete primitive in this case is $y = e^{cx - c^2}$, and $y = e^{\frac{1}{4}x^2}$ is a proper envelop, but the axis, $y = 0$, is not an envelop, but the particular integral corresponding to $c = \infty$.

7. It will be observed that in this example the expr. $\sqrt{[y^2(x^2 - 4 \log y)]}$ becomes imaginary when y passes through zero not by reason of the factor y^2 but because $\log y$ becomes imaginary when y becomes negative. We need only in fact observe the existence of the function $\log y$ in the differential equation to infer that $y = 0$ is a portion of the boundary line. So in the example

$$p = \frac{y \log y}{x}$$

we see at once that $y = 0$ is the boundary line, for p is imaginary when y is negative and real when y is positive. The complete primitive in this case is $y = e^{cx}$. When $c = +\infty$ the particular integral gives $y = 0$ if x is negative, that is coincides with one half of the axis of x, while if $c = -\infty$ the particular integral coincides with the other half of the axis. Thus the boundary is made up of portions of two of the particular integrals.

8. If the equation is of the first degree with respest to p, and algebraic with respect to x and y, p will be possible for every point of the plane and there will be no singular solution. If the equation is algebraic and of a degree higher than the second with respect to p we may apply the usual condition for equal roots that is to say, the equations $\varphi(x, y, p) = 0$ and $\varphi'(x, y, p) = 0$, where $\varphi' = \frac{d\varphi}{dp}$ must be satisfied by a common value of p: hence eliminating p between these equations we have the condition expressed as a relation between x and y. For example, given

$$\varphi = p^3 - 4xyp + 8y^2 = 0,$$

then

$$\varphi' = 3p^2 - 4xy = 0;$$

eliminating p we find $y = 0$, and $27y = 4x^3$.

Each of these is a branch of the envelop of the complete primitive, and is a singular solution. The complete primitive in fact is $y = c(x - c)^2$, representing a series of parabolas which touch the axis of x and the cubical parabola $27y = 4x^3$.

SOME TRIGONOMETRIC SERIES.

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1. Take the equations $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4, \ldots$ (1)

$$\log(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4, \ldots \quad (2)$$

$$\theta\sqrt{-1} = \varphi,\ 2\cos\theta = e^{\phi} + e^{-\phi},\ 2\sqrt{-1}.\sin\theta = e^{\phi} - e^{-\phi}, \ldots \quad (3)$$

$$\log 2\cos\theta = \varphi + \log(1 + e^{-2\phi}) = -\varphi + \log(1 + e^{2\phi}), \ldots \quad (4)$$

$$\log 2\sin\theta = -\log\sqrt{-1} + \varphi + \log(1 - e^{-2\phi}) = \log\sqrt{-1} - \varphi + \log(1 - e^{2\phi}), \quad (5)$$

$$2\log 2\cos\theta = \log(1 + e^{2\phi}) + \log(1 + e^{-2\phi}), \ldots \quad (6)$$

$$2\log 2\sin\theta = \log(1 - e^{2\phi}) + \log(1 - e^{-2\phi}). \ldots \quad (7)$$

Developing (6) and (7) by (1) and (2), we have

$$2\log 2\cos\theta = e^{2\phi} + e^{-2\phi} - \tfrac{1}{2}(e^{4\phi} + e^{-4\phi}) + \tfrac{1}{3}(e^{6\phi} + e^{-6\phi}), \ldots (8)$$

$$2\log 2\sin\theta = -(e^{2\phi} + e^{-2\phi}) - \tfrac{1}{2}(e^{4\phi} + e^{-4\phi}) - \tfrac{1}{3}(e^{6\phi} + e^{-6\phi}). \ldots (9)$$

Whence we have $\log\ 2\cos\theta = \cos 2\theta - \frac{1}{2}\cos 4\theta + \frac{1}{3}\cos 6\theta -, \ldots$ (10)

$$\log(2\sin\theta)^{-1} = \cos 2\theta + \tfrac{1}{2}\cos 4\theta + \tfrac{1}{3}\cos 6\theta + \ldots (11)$$

By differentiating (10) we have

$$\tan\theta = 2[\sin 2\theta - \sin 4\theta + \sin 6\theta - \ldots] \ldots (12)$$